Arithmetically Cohen-Macaulay Curves cut out by Quadrics

Sheldon Katz
Department of Mathematics
Oklahoma State University
Stillwater, OK 74078

The following question was raised by M. Stillman.

Main Question: Let $C \subset \mathbf{P}^r = \mathbf{CP}^r$ be a smooth arithmetically Cohen-Macaulay curve which is cut out scheme theoretically by quadrics. Is the homogeneous ideal of C necessarily cut out by quadrics?

In [4], it was shown that the question has an affirmative answer if $r \leq 5$. The purpose of this note is to show that the question has a negative answer (there is a counterexample with $r = 7$).

It is a pleasure to thank D. Eisenbud, M. Stillman, and C. Walter for helpful conversations.

1 Homogeneous and Scheme-Theoretic Generation by Quadrics

Let X be a projective variety. It is often of interest to know whether or not the homogeneous ideal of X can be generated by quadrics, e.g. if X is a general canonical curve. In such a case, X is cut out scheme-theoretically by quadrics as well. It is usually easier to verify the scheme-theoretic statement— this amounts to ignoring the vertex of the affine cone over X.

Problem: Let $C \subset \mathbf{P}^r = \mathbf{CP}^r$ be a smooth curve which is cut out scheme theoretically by quadrics. Is the homogeneous ideal of C necessarily cut out by quadrics?

In [4], this problem was investigated. The answer is a resounding <u>no</u>. A counterexample was found with $r = 5$. However, positive results were found. The problem has an affirmative answer for curves on scrolls, all curves with

$r \leq 4$, and arithmetically Cohen-Macaulay curves which lie on projectively normal K3 surfaces cut out by quadrics (this includes all arithmetically C-M curves with $r = 5$). This leads to a more precise question, which we could not answer:

Question: Let $C \subset \mathbf{P}^r = \mathbf{CP}^r$ be a smooth arithmetically Cohen-Macaulay curve which is cut out scheme theoretically by quadrics. Is the homogeneous ideal of C necessarily cut out by quadrics?

It turns out that this question also has a negative answer.

Proposition 1 *Let $C \subset \mathbf{P}^7$ be a general degree 19 embedding of a general genus 12 curve over an algebraically closed field of characteristic 0. Then C is smooth and arithmetically Cohen-Macaulay, C is cut out scheme-theoretically by quadrics, and the homogeneous ideal of C is not cut out by quadrics.*

2 Candidates for a Counterexample

Let $C \subset \mathbf{P}^r$ be an arithmetically Cohen-Macaulay curve of degree d and genus g. Assume in addition that $\mathcal{O}_C(1)$ is non-special, i.e. $H^1(\mathcal{O}_C(1)) = 0$. Then $d = g + r$.

It turns out that for certain values of g and r, the homogeneous ideal of such a curve C *cannot* be cut out by quadrics, for simple dimension reasons. Let I denote the ideal sheaf of C. Then if

$$(r+1)h^0(I(2)) < h^0(I(3)), \tag{1}$$

the natural map

$$H^0(I(2)) \otimes H^0(\mathcal{O}_{\mathbf{P}^r}(1)) \to H^0(I(3))$$

cannot be surjective, so that the homogeneous ideal of C cannot be generated by quadrics. Using Riemann-Roch, (1) becomes

$$g > \frac{r(r-2)}{3}. \tag{2}$$

On the other hand, if we want C to be scheme-theoretically cut out by quadrics, then we must have enough quadrics, i.e.

$$\binom{r}{2} - g \geq r - 1.$$

Equality holds if and only if C is a complete intersection of $r-1$ quadrics; but in this case the homogeneous ideal is cut out by quadrics as well. This can be improved slightly: in [4, Cor. 2.5] it was shown that if C is cut out scheme theoretically by r quadrics, then necessarily

$$g = (r-1)d/2 + 1 - 2^{r-1}. \tag{3}$$

So if (3) does not hold, then

$$g \leq \frac{r^2 - 3r - 2}{2} \tag{4}$$

There are no counterexamples to the main question for $r \leq 5$ [4]. Suppose that there is a non-special counterexample with $r = 6$. Then $g \geq 9$ by (2). Since (4) gives $g \leq 8$, it follows that (3) holds, and $g = 9$. But then $d = 15$, and a contradiction is reached.

Turning next to $r = 7$, (2) gives $g \geq 12$, and (4) gives $g \leq 13$. In the following section, we show that in fact the *general* curve of degree 19 and genus 12 in $\mathbf{P}^7$ is a counterexample.

3 The counterexample

Pick 22 general points $p_1, p_2, p_3, q_1, \ldots, q_7, r_1, \ldots, r_{12}$ in $\mathbf{P}^2$. Let C' be a general plane curve of degree 9 passing through the p_i with multiplicity 3, through the q_i with multiplicity 2, and simply through the r_i. The linear system $|L|$ of degree 7 curves passing doubly through the p_i and simply through the q_i and r_i maps C' birationally to a smooth curve C of degree 19 and arithmetic genus 12 in P^7.

It is a simple matter to use Macaulay [3] to construct such a curve. In describing the calculation, I will informally say that a general curve has a certain property, when I mean that the property is satisfied for an example curve constructed using Macaulay's pseudo-random number generator. In fact, I repeated the construction several times with different pseudo-random coefficients, and the properties mentioned below held in each instance. Thus, as expected, a "general" curve has been constructed.

MACAULAY's pseudo-random number generator is used to construct 22 "general" points in $\mathbf{P}^2{}_{\mathbf{F}_{31991}}$, and from this the curve C' (actually, there is no harm in supposing that the p_i are $(1,0,0),(0,1,0),(0,0,1)$, to shorten computations). By calculating the Jacobian of C', it is checked that the singular scheme of C' has degree 19 as expected (triple points count at least 4 times). Hence C' has the expected geometric genus 12. The equations of the image curve C can then be explicitly calculated. C is cut out ideal theoretically by 9 independent quadrics and 2 independent cubics, and has Hilbert function $(1+6t+12t^2)(1-t)^{-2}$. In particular C has arithmetic genus 12; being the image of the normalization of C' by the base point free system $|L|$ on the blowup of $\mathbf{P}^2$, it follows that C is smooth. Let $\tilde{C} \subset \mathbf{P}^2$ be the scheme cut out by the 9 quadrics alone. Via MACAULAY, $\tilde{C}$ has degree 19 and arithmetic genus 12. It follows easily that $C = \tilde{C}$, i.e. C is cut out scheme-theoretically by quadrics.

Next, to see that C is arithmetically Cohen-Macaulay, note that C is non-special since the projective dimension of the embedding system is 7, is linearly normal by construction, and is quadratically normal by Riemann-Roch and $h^0(I_C(2)) = 9$ found by MACAULAY. This suffices to show that C is arithmetically Cohen-Macaulay by [1, P. 222] or the argument in the proof of Theorem 1.2.7 in [8].

Proof of Proposition 1: The key point is to show that the conditions "arithmetically Cohen-Macaulay" and "scheme-theoretically cut out by quadrics" are dense.

A curve is arithmetically Cohen-Macaulay if and only if it is projectively normal. So C is arithmetically Cohen-Macaulay if and only if $H^1(I_C(n)) = 0$ for all $n \geq 0$, where I_C is the ideal sheaf of C. By [7], $H^1(I_C(n)) = 0$ for all $n \geq 13$, so there are only finitely many cohomology groups that are required to vanish in addition. By upper semicontinuity of $h^1(I_C(n)) = \dim\ H^1(I_C(n))$, this is a Zariski open condition in the Hilbert scheme.

As to the condition of being scheme-theoretically cut out by quadrics, we may restrict to considering curves which are arithmetically Cohen-Macaulay. Let V be the 9 dimensional space of quadrics containing C. Consider the maps

$$V \otimes H^0(\mathcal{O}_{\mathbf{P}^7}(k)) \to H^0(I_C(k+2)) \tag{5}$$

V cuts out C scheme-theoretically if and only if (5) is surjective for some

$k \geq 12$ (since C is 14-regular by [7]; a smaller bound for effective k can be given if desired). This is again an open condition.

Finally, let Hilb^0_{19n-11} be the subset of the Hilbert scheme parametrizing smooth, irreducible curves in $\mathbf{P}^7$ of degree 19 and genus 12. It is open in the Hilbert scheme by [9, P. 99]. Hilb^0_{19n-11} is defined over Spec $\mathbf{Z}$ and is irreducible (its geometric fibers are equidimensional and irreducible; this follows from the irreducibility of $\mathcal{M}_{12}$ in arbitrary characteristic [5], and the non-speciality of $|L|$).

Hence the set of smooth arithmetically Cohen-Macaulay curves scheme-theoretically cut out by quadrics is non-empty and open, hence dense, in Hilb^0_{19n-11}. This completes the proof of Proposition 1.

QED

It seems appropriate to conclude with some related questions.

In [1], [6, §3], it was proven that a general linear system of degree $d \geq [(3g+4)/2]$ on a curve C of genus g embeds C as an arithmetically Cohen-Macaulay curve. Rather than looking for a bound for *all* curves, instead one can ask:

Problem: Find the smallest possible $d(g)$ such that for all $d \geq d(g)$, a general curve of genus g admits a degree d complete embedding which is arithmetically Cohen-Macaulay.

Remark. Suppose that $d \geq (2g+1+\sqrt{8g+1})/2$. Then the general degree d embedding of a general curve of genus g is arithmetically Cohen-Macaulay [2]. This bound is in fact sharp for *non-special* embeddings. The inequality is just the solution of the inequality $h^0(\mathcal{O}_{\mathbf{P}^r}(2)) \geq h^0(\mathcal{O}_C(2))$ for a general non-special embedding.

Similarly, one can ask

Problem: Find the smallest possible $d'(g)$ such that a general degree d embedding of a general curve of genus g is scheme theoretically cut out by quadrics if $d \geq d'(g)$.

By work of Green and Lazarsfeld [8, Prop. 2.4.2], $d'(g) \leq [(3g+6)/2]$, and Proposition 1 shows that this is not sharp.

Question: Is Proposition 1 true without restriction on the characteristic? Is there a counterexample to the main question with $r = 6$?

References

[1] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris. Geometry of Algebraic Curves, Vol I, Springer 1985.

[2] E. Ballico and Ph. Ellia. On postulation of curves: Embeddings by complete linear series. Arch. Math. **43** (1984) 244–249.

[3] D. Bayer and M. Stillman. Macaulay, a computer algebra system. Available free from the authors for the Macintosh, VAX, SUN, and many other computers (1989) (with scripts by D. Eisenbud and M. Stillman).

[4] L. Ein, D. Eisenbud, and S. Katz. Varieties cut out by quadrics: Scheme-theoretic versus homogeneous generation of ideals. In: Algebraic Geometry, Sundance 1986, SLN 1311.

[5] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. Publ. Math. IHES **36** (1969), 75–110.

[6] M. Green and R. Lazarsfeld. On the projective normality of complete linear series on an algebraic curve. Inv. Math. **83** (1986) 73–90.

[7] L. Gruson, R. Lazarsfeld, and C. Peskine. On a theorem of Castelnuovo, and the equations defining space curves. Inv. math. **72** (1983) 491–506.

[8] R. Lazarsfeld. A sampling of vector bundle techniques in the study of linear series. Informal note for lectures at ICTP College on Riemann Surfaces, Trieste, December 1987.

[9] D. Mumford. Geometric Invariant Theory. Springer-Verlag, Heidelberg-Berlin-New York, 1965.